On the Fractional-Order Logistic Equation with Two Different Delays

Ahmed M. A. El-Sayed[a], Hala A. A. El-Saka[b], and Esam M. El-Maghrabi[c]

[a] Faculty of Science, Alexandria University, Alexandria, Egypt
[b] Department of Mathematics, Damietta Faculty of Science, 34517, New Damietta, Egypt
[c] Faculty of Science, Benha University, Benha 13518, Egypt

Reprint requests to H. A. A. E.-S.; E-mail: halaelsaka@yahoo.com

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The fractional-order logistic equation with the two different delays $r_1, r_2 > 0$, $\mathrm{D}^{\alpha}x(t) = \rho x(t - r_1)[1 - x(t - r_2)]$, $t > 0$ and $\rho > 0$, with the initial data $x(t) = x_0, t \leq 0$ are considered. The existence of a unique uniformly stable solution is studied and the Adams-type predictor-corrector method is applied to obtain the numerical solution.

Key words: Logistic Delay Equation; Fractional-Order Differential Equations; Stability; Existence; Uniqueness; Numerical Solution; Predictor-Corrector Method.

1. Introduction

The topic of fractional calculus (derivatives and integrals of arbitrary orders) is enjoying growing interest not only among mathematicians, but also among physicists and engineers.

Let $I = [0,T]$, $T < \infty$; $C(I)$ is the class of all continuous functions defined on I with norm $\|x\| = \sup_t |\mathrm{e}^{-Nt}x(t)|, N > 0$; $L_1[0,T] = L_1$ is the class of all integrable functions on I with the norm $\|x\|_1 = \int_0^T \mathrm{e}^{-Nt}|x(t)|\mathrm{d}t, N > 0$.

Let $\alpha \in (0,1]$. Here we are concerned with the initial value problem of the fractional-order logistic equation with the two different delays r_1 and r_2:

$$\mathrm{D}^{\alpha}x(t) = \rho x(t - r_1)[1 - x(t - r_2)], \quad t > 0, \quad (1)$$

$$x(t) = x_0, \quad t \leq 0. \quad (2)$$

In Section 2 we study the existence and uniqueness of the solution. In Section 3 we study the stability of the solution, and in Section 4 we apply the predict-evaluate-correct-evaluate (PECE) method to obtain the numerical solution.

Now, we give the definition of fractional-order integration and fractional-order differentiation.

Definition. The fractional integral of order $\beta \in \mathbb{R}^+$ of the function $f(t)$, $t \in I$, is

$$I^{\beta} f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\mathrm{d}s.$$

Definition. The (Caputo) fractional-order derivative D^{α} of order $\alpha \in (0,1]$ of the function $f(t)$ is given by

$$\mathrm{D}^{\alpha} f(t) = I^{1-\alpha}$$

2. Existence and Uniqueness

For the initial value problem (1) – (2) define $C(I) = \{x \in \mathbb{R} : x(t) \in [0,1], t \in I \text{ and } x(t) = x_0, t \leq 0\}$.

Theorem 1. The initial value problem (1) – (2) has a unique solution $x \in C(I)$, $x' \in L_1$.

Proof. From the properties of fractional calculus the fractional-order differential equation in (1) can be written as

$$I^{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \rho x(t - r_1)[1 - x(t - r_2)];$$

operating with I^{α}, we obtain

$$x(t) = x_0 + \rho I^{\alpha} x(t - r_1)[1 - x(t - r_2)]. \quad (3)$$

Now let the operator $F : C(I) \to C(I)$ be defined by

$$Fx(t) = x_0 + \rho I^{\alpha}[x(t - r_1) - x(t - r_1)x(t - r_2)]. \quad (4)$$

Then

$$\mathrm{e}^{-Nt}|Fx - Fy| \leq$$
$$2\rho \int_0^{r_1} \mathrm{e}^{-Nt} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x(s - r_1) - y(s - r_1)|\mathrm{d}s$$

$$+2\rho\int_{r_1}^{t}\mathrm{e}^{-Nt}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|x(s-r_1)-y(s-r_1)|\mathrm{d}s$$
$$+\rho\int_{0}^{r_2}\mathrm{e}^{-Nt}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|x(s-r_2)-y(s-r_2)|\mathrm{d}s$$
$$+\rho\int_{r_2}^{t}\mathrm{e}^{-Nt}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|x(s-r_2)-y(s-r_2)|\mathrm{d}s,$$

but $x(t)=x_0$ and $y(t)=x_0$ when $t\le 0$, then

$$\mathrm{e}^{-Nt}|Fx-Fy|\le$$
$$2\rho\int_{0}^{t-r_1}\mathrm{e}^{-N(t-\theta)}\frac{(t-r_1-\theta)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{e}^{-N\theta}|x(\theta)-y(\theta)|\mathrm{d}\theta$$
$$+\rho\int_{0}^{t-r_2}\mathrm{e}^{-N(t-\theta)}\frac{(t-r_2-\theta)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{e}^{-N\theta}|x(\theta)-y(\theta)|\mathrm{d}\theta.$$

This implies that

$$\|Fx-Fy\|\le\frac{3\rho}{N^{\alpha}}\|x-y\|,$$

and it can be proved that if we choose N large enough such that $N^{\alpha}>3\rho$, we obtain

$$\|Fx-Fy\|<\|x-y\|,$$

and the operator F has a unique fixed point $x\in C(I)$.

Now from (3), we have

$$x(t)=x_0+\rho x_0\int_{0}^{r_1}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s$$
$$+\rho\int_{r_1}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s-r_1)\mathrm{d}s$$
$$-\rho x_0^2\int_{0}^{r_1}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s$$
$$-\rho x_0\int_{r_1}^{r_2}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s-r_1)\mathrm{d}s$$
$$-\rho\int_{r_2}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s-r_1)x(s-r_2)\mathrm{d}s.$$

Differentiating formally, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t)=\rho(x_0-x_0^2)\frac{t^{\alpha-1}}{\Gamma(\alpha)}$$
$$+\rho\int_{r_1}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x'(s-r_1)\mathrm{d}s$$
$$-\rho x_0\int_{r_1}^{r_2}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x'(s-r_1)\mathrm{d}s$$
$$-\rho\int_{r_2}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x'(s-r_1)x(s-r_2)\mathrm{d}s$$
$$-\rho\int_{r_2}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s-r_1)x'(s-r_2)\mathrm{d}s,$$

which proves that

$$\|x'\|\le\frac{\rho}{N^{\alpha}}|x_0-x_0^2|+\frac{3\rho}{N^{\alpha}}\|x'\|,$$

i. e.

$$\|x'\|\le\left(1-\frac{3\rho}{N^{\alpha}}\right)^{-1}\frac{\rho|x_0-x_0^2|}{N^{\alpha}},$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t)\in L_1.$$

And from (3), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t)=\rho\frac{\mathrm{d}}{\mathrm{d}t}I^{\alpha}x(t-r_1)[1-x(t-r_2)],$$

operating with $I^{1-\alpha}$, we obtain (1) and the theorem is proved.

3. Stability of the Solution

Let $x(t)$ be a solution of the initial value problem (1)–(2) and let $x^*(t)$ be a solution of

$$\mathrm{D}^{\alpha}x^*(t)=\rho x^*(t-r_1)[1-x^*(t-r_2)],\quad t>0,$$
$$x^*(t)=x_0^*,\quad t\le 0, \tag{5}$$

then we get

$$x(t)-x^*(t)=(x_0-x_0^*)+\rho I^{\alpha}\Big\{x(t-r_1)-x^*(t-r_1)$$
$$+x(t-r_1)\big[x^*(t-r_2)-x(t-r_2)\big]$$
$$+x^*(t-r_2)\big[x^*(t-r_1)-x(t-r_1)\big]\Big\}$$

and

$$\mathrm{e}^{-Nt}|x(t)-x^*(t)|\le\mathrm{e}^{-Nt}|x_0-x_0^*|$$
$$+2\rho\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{(t-r_1)^{\alpha}}{\Gamma(\alpha+1)}\right]|x_0-x_0^*|$$
$$+2\rho\int_{0}^{t-r_1}\mathrm{e}^{-N(t-\theta)}\frac{(t-r_1-\theta)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{e}^{-N\theta}|x(\theta)-x^*(\theta)|\mathrm{d}\theta$$
$$+\rho\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{(t-r_2)^{\alpha}}{\Gamma(\alpha+1)}\right]|x_0-x_0^*|$$
$$+\rho\int_{0}^{t-r_2}\mathrm{e}^{-N(t-\theta)}\frac{(t-r_2-\theta)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{e}^{-N\theta}|x(\theta)-x^*(\theta)|\mathrm{d}\theta,$$

which gives

$$\|x-x^*\| \le \left(1+\frac{3\rho T^\alpha}{\Gamma(\alpha+1)}\right)|x_0-x_0^*| + \frac{3\rho}{N^\alpha}\|x-x^*\|,$$

i. e.

$$\|x-x^*\| \le \left(1-\frac{3\rho}{N^\alpha}\right)^{-1}\left(1+\frac{3\rho T^\alpha}{\Gamma(\alpha+1)}\right)|x_0-x_0^*|,$$

therefore if $|x_0-x_0^*| < \delta(\varepsilon)$ implies $\|x-x^*\| < \varepsilon$. Then the solution of (1) – (2) is uniformly stable.

4. Numerical Methods and Results

An Adams-type predictor-corrector method has been introduced in [4, 5] and investigated further in [1 – 3, 6 – 10]. In this paper, we use an Adams-type predictor-corrector method for the numerical solution of fractional integral equation.

The key to the derivation of the method is to replace the original fractional differential equation in

$$\mathrm{D}^\alpha x(t) = f(x(t))$$

by the fractional integral equation

$$x(t) = x_0 + I^\alpha f(x(t)). \tag{6}$$

The product trapezoidal quadrature formula is used with nodes t_j $(j = 0, 1, \ldots, k+1)$ taken with respect to the weight function $(t_{k+1} - .)^{\alpha-1}$. In other words, they applied the approximation

$$\int_{t_0}^{t_{k+1}} (t_{k+1}-u)^{\alpha-1} g(u)\mathrm{d}u \approx \int_{t_0}^{t_{k+1}} (t_{k+1}-u)^{\alpha-1} g_{k+1}(u)\mathrm{d}u = \sum_{j=0}^{k+1} a_{j,k+1} g(t_j),$$

where

$$a_{j,k+1} = \begin{cases} \dfrac{h^\alpha}{\alpha(\alpha+1)}[k^{\alpha+1} - (k-\alpha)(k+1)^\alpha], & \text{if } j = 0, \\ \dfrac{h^\alpha}{\alpha(\alpha+1)}, & \text{if } j = k+1. \end{cases}$$

h is a step size, and for $1 \le j \le k$ holds

$$a_{j,k+1} = \frac{h^\alpha}{\alpha(\alpha+1)}\Big[(k-j+2)^{\alpha+1} - 2(k-j+1)^{\alpha+1} + (k-j)^{\alpha+1}\Big].$$

This yield the corrector formula, i. e. the fractional variant of the one-step Adams-Moulton method

$$x_{k+1} = x_0 + \frac{1}{\Gamma(\alpha)}\left(\sum_{j=0}^{k} a_{j,k+1} f(x_j) + a_{k+1,k+1} f(x_{k+1}^p)\right). \tag{7}$$

The remaining problem is the determination of the predictor formula that is needed to calculate the value x_{k+1}^p. The idea they used to generalize the one-step Adams-Bashforth method is the same as the one described above for the Adams-Moulton technique: the integral on the right-hand side of (6) is replaced by the product rectangle rule, i. e.

$$\int_{t_0}^{t_{k+1}} (t_{k+1}-u)^{\alpha-1} g(u)\mathrm{d}u \approx \sum_{j=0}^{k} b_{j,k+1} g(t_j),$$

where

$$b_{j,k+1} = \frac{h^\alpha}{\alpha}\left[(k+1-j)^\alpha - (k-j)^\alpha\right].$$

Thus, the predictor x_{k+1}^p is determined by the fractional Adams-Bashforth method

$$x_{k+1}^p = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k} b_{j,k+1} f(x_j). \tag{8}$$

This completes the description of the basic algorithm, namely, the fractional version of the one-step Adams-Bashforth Moulton method. Recapitulating, they saw that they first calculated the predictor x_{k+1}^p according to (8), then they evaluated $f(x_{k+1}^p)$, used this to determine the corrector x_{k+1} by means of (7), and finally evaluated $f(x_{k+1})$ which is then used in the next integration step. Methods of this type are usually called predictor-corrector or, more precisely, predict-evaluate-correct-evaluate (PECE) methods.

Now, we apply the PECE method to the problem (1) – (2).

The approximate solutions are displayed in Figures 1 – 3 for different x_0 and α. In Figure 1, we take $\rho = 0.5$, $r_1 = r_2 = 0.7$, $\alpha = 0.9$, and different x_0. In Figure 2, we take $\rho = 0.5$, $r_1 = 0.2$, $r_2 = 0.7$, $x_0 = 0.85$, and different α. In Figure 3, we take $\rho = 0.5$, $r_1 = r_2 = 0$, $x_0 = 0.85$, and different α.

These figures assure that the solution of (1) – (2) is uniformly stable. In Figure 3 with $r_1 = r_2 = 0$, the same results as in [13] are obtained.

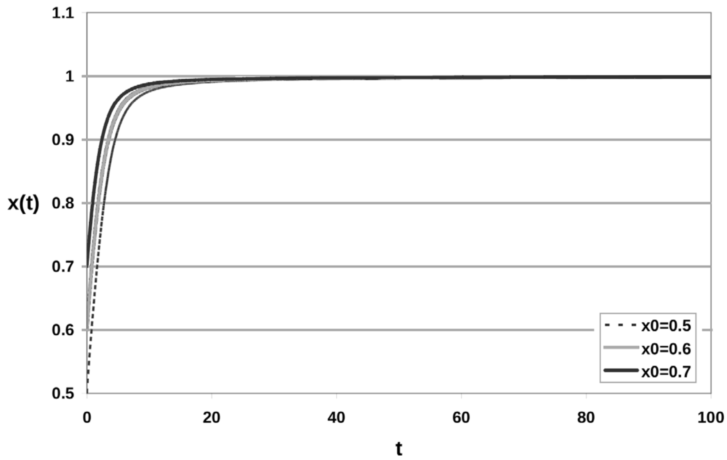

Fig. 1. Approximate solution for $\rho = 0.5$, $r_1 = r_2 = 0.7$, $\alpha = 0.9$, and different x_0.

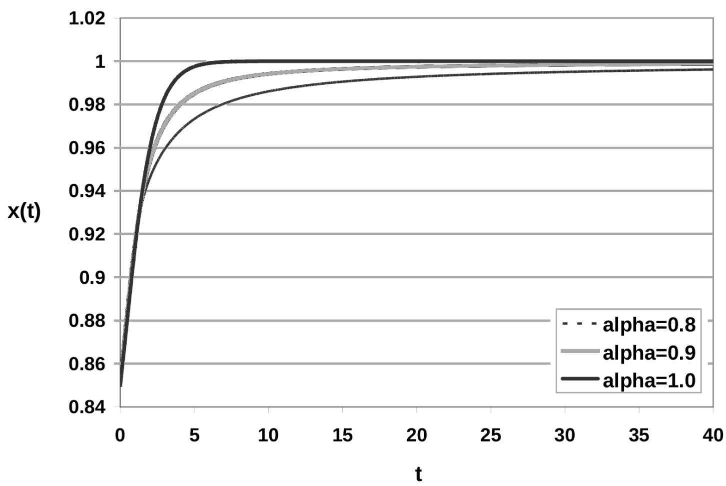

Fig. 2. Approximate solution for $\rho = 0.5$, $r_1 = 0.2$, $r_2 = 0.7$, $x_0 = 0.85$, and different α.

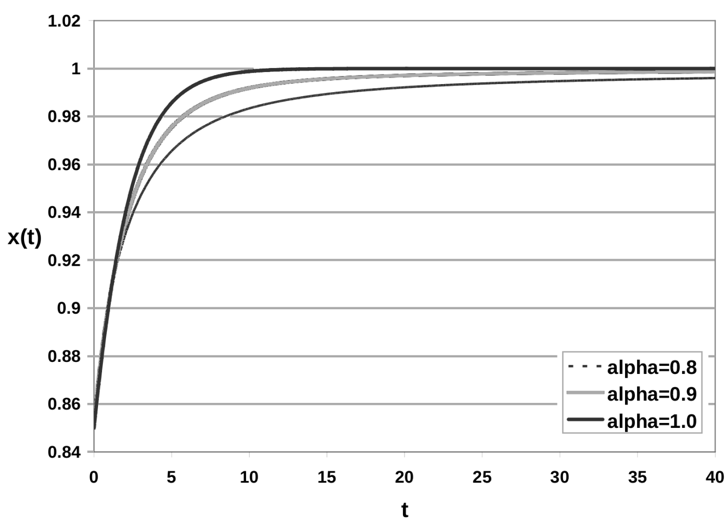

Fig. 3. Approximate solution for $\rho = 0.5$, $r_1 = r_2 = 0$, $x_0 = 0.85$, and different α.

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